Spin matrix elements in 2D Ising model on the finite lattice

A. I. Bugrij *, **O. Lisovyy** *, †

* Bogolyubov Institute for Theoretical Physics
Metrolohichna str., 14-b, Kyiv-143, 03143, Ukraine

† Laboratoire de Mathématiques et Physique Théorique CNRS/UMR 6083,
Université de Tours, Parc de Grandmont, 37200 Tours, France

Abstract

We present explicit formulas for all spin matrix elements in the 2D Ising model with the nearest neighbor interaction on the finite periodic square lattice. These expressions generalize the known results [1, 3, 6] (coincide with them in the appropriate limits) and fulfill the test of straightforward transfer matrix calculations for finite N.

1 Eigenvalues and eigenvectors of the transfer matrix

It is well-known (see [7, 8]) that the spectrum of $2^N \times 2^N$ transfer matrix, corresponding to Ising model on the periodic square lattice, consists of two sets:

$$\lambda = (2s)^{N/2} \exp \left\{ \frac{1}{2} \left(\pm\gamma\left(0\right) \pm \gamma(2\pi/N) \pm \ldots \pm \gamma(2\pi - 2\pi/N) \right) \right\}, \tag{1}$$

$$\lambda = (2s)^{N/2} \exp \left\{ \frac{1}{2} (\pm\gamma(\pi/N) \pm \gamma(3\pi/N) \pm \ldots \pm \gamma(2\pi - \pi/N)) \right\}, \tag{2}$$

where $s = \sinh 2\mathcal{K}$ and $\mathcal{K}$ is the Ising coupling constant. The function $\gamma(q)$ is defined as the positive root of the equation

$$\cosh \gamma(q) = s + s^{-1} - \cos q.$$

which is the lattice analog of the relativistic energy dispersion law. The number of minuses in (1) is even in ferromagnetic ($s > 1$) and odd in paramagnetic ($0 < s < 1$) phase, while the number of minuses in (2) is even in both phases. The eigenvalues (1) (or (2)) correspond to eigenvectors that are odd (resp. even) under spin reflection.

The notation and terminology, introduced in [6] for the analysis of continuum limit, are also very convenient on the lattice. In what follows, odd and even eigenvectors of the Ising transfer matrix will be interpreted as multiparticle states from the Ramond and Neveu-Schwartz sector. Quasimomenta of R–particles can be equal to $\frac{2\pi}{N}j$ ($j = 0, 1, \ldots, N-1$), while for NS–particles they take on the values $\frac{2\pi}{N}\left(j + \frac{1}{2}\right)$ ($j = 0, 1, \ldots, N-1$). Each eigenstate consists of particles of only one type, and their quasimomenta must be different.

We will denote by $|p_1, \ldots, p_K\rangle_{NS(R)}$ the normalized eigenstate, containing particles with the momenta $p_1, \ldots, p_K$. Since R–sector in paramagnetic phase contains the state $|0\rangle_R$ (one particle with zero momentum), it will be convenient to denote NS and R vacua by $|\emptyset\rangle_{NS}$ and $|\emptyset\rangle_R$. The goal of the present paper is to find matrix elements ${}_{NS}\langle p_1, \ldots, p_K|\sigma|q_1, \ldots, q_L\rangle_R$ of the Ising spin σ in the described basis of normalized eigenstates. (R–R and NS–NS matrix elements vanish due to $\mathbb{Z}_2$-symmetry of the model).

2 Lattice form factors and scaling limit

All n-point correlation functions in the Ising model on the cylinder and torus can be easily expressed via spin matrix elements. However, known results were obtained in rather inverse way. At the first stage, 2-point functions are expressed through the determinants of certain Toeplitz matrices with a size that depends on the separation of correlating spins. To extract the analytic dependence on the distance from these representations, a lot of further work was needed [2]. The final answer [3, 4] allows to calculate squared form factors on the cylinder (on the infinite lattice the above program was realized earlier in [9, 10, 12]):

$$\left|{}_{NS}\langle\emptyset|\sigma|q_1,\ldots,q_L\rangle_R\right|^2 = \xi\,\xi_T \prod_{j=1}^{L} \frac{e^{-\nu(q_j)}}{N\sinh\gamma(q_j)} \prod_{1\le i<j\le L} \left(\frac{\sin\frac{q_i-q_j}{2}}{\sinh\frac{\gamma(q_i)+\gamma(q_j)}{2}}\right)^2. \tag{3}$$

Here $\xi = \left|1-s^{-4}\right|^{1/4}$, quasimomenta have discrete R–values and cylindrical parameters ξ_T, $\nu(q)$ are given by

$$\ln\xi_T = \frac{N^2}{2\pi^2}\int_0^{\pi}\int_0^{\pi} \frac{dp\,dq\,\gamma'(p)\gamma'(q)}{\sinh\left(N\gamma(p)\right)\sinh\left(N\gamma(q)\right)} \ln\left|\frac{\sin\left((p+q)/2\right)}{\sin\left((p-q)/2\right)}\right|, \tag{4}$$

$$\nu(q) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{dp\,\sinh\gamma(q)}{\cosh\gamma(q)-\cos p} \ln\coth\left(N\gamma(p)/2\right). \tag{5}$$

In the thermodynamic limit $N\to\infty$ these parameters vanish ($\xi_T\to 1$, $\nu(q)\to 0$) and (3) transforms into the classical formula [10, 12]:

$$\left|{}_{NS}\langle\emptyset|\sigma|q_1,\ldots,q_L\rangle_R\right|^2 = \xi \prod_{j=1}^{L} \frac{1}{\sinh\gamma(q_j)} \prod_{1\le i<j\le L} \left(\frac{\sin\frac{q_i-q_j}{2}}{\sinh\frac{\gamma(q_i)+\gamma(q_j)}{2}}\right)^2, \tag{6}$$

where $\{q_j\}$ can take on arbitrary values in the interval $[-\pi,\pi\,]$.

In the scaling limit, Ising model on the plane was shown to be equivalent to a relativistic quantum field theory with two-particle S-matrix equal to -1 (see [11]). Then it became possible to use the results of [1] and to calculate all spin matrix elements:

$${}_{NS}\langle p_1,\ldots,p_K|\sigma|q_1,\ldots,q_L\rangle_R = \sqrt{\xi}\prod_{i=1}^{K}\frac{1}{\sqrt{2\pi\omega(p_i)}}\prod_{j=1}^{L}\frac{1}{\sqrt{2\pi\omega(q_j)}}\;F\Big(\{p\}\big|\{q\}\Big), \tag{7}$$

$$F\Big(\{p\}\big|\{q\}\Big) = \prod_{1\le i<j\le K}\frac{p_i-p_j}{\omega(p_i)+\omega(p_j)}\prod_{1\le i<j\le L}\frac{q_i-q_j}{\omega(q_i)+\omega(q_j)}\prod_{\substack{1\le i\le K\\ 1\le j\le L}}\frac{\omega(p_i)+\omega(q_j)}{p_i-q_j}. \tag{8}$$

Here $\omega(q)=\sqrt{m^2+q^2}$ and the momenta of both type of particles take on arbitrary real values. The RHS of (7) is usually written with the factor $i^{[\frac{K+L}{2}]}$, but it can be removed by a change of the basis and will be omitted in what follows.

Very recently, Fonseca and Zamolodchikov [6] announced and promised to give a proof of a similar formula for the scaling limit on the cylinder:

$${}_{NS}\langle p_1,\ldots,p_K|\sigma|q_1,\ldots,q_L\rangle_R = \sqrt{\xi\,\tilde{\xi}_T}\prod_{i=1}^{K}\frac{e^{\tilde{\nu}(p_i)/2}}{\sqrt{\beta\,\omega(p_i)}}\prod_{j=1}^{L}\frac{e^{-\tilde{\nu}(q_j)/2}}{\sqrt{\beta\,\omega(q_j)}}\;F\Big(\{p\}\big|\{q\}\Big). \tag{9}$$

The overall factor $\tilde{\xi}_T$ and the function $\tilde{\nu}(q)$ from the leg factors are determined from the scaling limit of (4), (5):

$$\ln \tilde{\xi}_T = \frac{m^2\beta^2}{2\pi^2} \int_0^\infty \int_0^\infty \frac{dp\, dq\, \omega'(p)\omega'(q)}{\sinh(\beta\,\omega(p))\sinh(\beta\,\omega(q))} \ln\left|\frac{p+q}{p-q}\right|, \tag{10}$$

$$\tilde{\nu}(q) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dp\ \omega(q)}{p^2+q^2+m^2}\ \ln\coth\frac{\beta\omega(p)}{2}. \tag{11}$$

Here, β denotes the scaled length of the base of the cylinder, NS–momenta are quantized as $p_j = \frac{2\pi}{\beta}\, l_j$, $l_j \in \mathbb{Z}+\frac{1}{2}$, while for R–momenta we have $q_j = \frac{2\pi}{\beta}\, l_j$ and $l_j \in \mathbb{Z}$.

3 General formula

On the level of form factors ${}_{NS}\langle\emptyset|\sigma|\{q\}\rangle_R$ the expression (9) represents nothing new and can even be proven rigorously — it is simply a particular case of (3). However, this conjecture gives *all* matrix elements, though only in the scaling limit. Moreover, the structure of this representation is so transparent that the lattice generalization immediately suggests itself. To be more precise, having taken into account the correspondence between (3) and (9), we propose the following general formula for spin matrix elements on finite periodic lattice:

$$\begin{aligned}{}_{NS}\langle p_1,\ldots,p_K|\sigma|q_1,\ldots,q_L\rangle_R &= \sqrt{\xi\,\xi_T}\prod_{i=1}^{K}\frac{e^{\nu(p_i)/2}}{\sqrt{N\sinh\gamma(p_i)}}\prod_{j=1}^{L}\frac{e^{-\nu(q_j)/2}}{\sqrt{N\sinh\gamma(q_j)}}\times\\ &\times\prod_{1\le i<j\le K}\frac{\sin\frac{p_i-p_j}{2}}{\sinh\frac{\gamma(p_i)+\gamma(p_j)}{2}}\prod_{1\le i<j\le L}\frac{\sin\frac{q_i-q_j}{2}}{\sinh\frac{\gamma(q_i)+\gamma(q_j)}{2}}\prod_{\substack{1\le i\le K\\ 1\le j\le L}}\frac{\sinh\frac{\gamma(p_i)+\gamma(q_j)}{2}}{\sin\frac{p_i-q_j}{2}},\end{aligned} \tag{12}$$

where ξ_T, $\nu(q)$ are defined by (4) and (5). All previous results can be easily obtained from this expression in appropriate limits. However, we have not yet found a rigorous proof of (12). Instead, since this formula should hold even on the finite lattice, we have verified it explicitly for small N.

As an illustration, let us consider 3-row Ising chain in the ferromagnetic region of temperature parameter ($s>1$). In this case NS (R) momenta take on the values π, $\pi/3$, $-\pi/3$ (0, $2\pi/3$, $-2\pi/3$). Each state contains either two particles or no particles at all. Since the integrals (4) and (5) can be alternatively written as

$$\xi_T^4 = \frac{\prod_q^{(R)}\prod_p^{(NS)}\sinh^2\frac{\gamma(q)+\gamma(p)}{2}}{\prod_q^{(R)}\prod_p^{(R)}\sinh\frac{\gamma(q)+\gamma(p)}{2}\prod_q^{(NS)}\prod_p^{(NS)}\sinh\frac{\gamma(q)+\gamma(p)}{2}},\quad \nu(q) = \ln\frac{\prod_p^{(NS)}\sinh\frac{\gamma(q)+\gamma(p)}{2}}{\prod_p^{(R)}\sinh\frac{\gamma(q)+\gamma(p)}{2}},$$

then to verify (12) it suffices to prove ten relations:

$$\begin{aligned}{}_{NS}\langle\emptyset|\sigma|\emptyset\rangle_R^{\,2} &= \frac{\sinh\frac{\gamma_0+\gamma_{\pi/3}}{2}\sinh\frac{\gamma_\pi+\gamma_{2\pi/3}}{2}\sinh^2\frac{\gamma_{\pi/3}+\gamma_{2\pi/3}}{2}}{\sinh\gamma_{2\pi/3}\sinh\gamma_{\pi/3}\sinh\frac{\gamma_0+\gamma_{2\pi/3}}{2}\sinh\frac{\gamma_\pi+\gamma_{\pi/3}}{2}},\\ {}_{NS}\langle-\pi/3,\pi/3|\sigma|2\pi/3,-2\pi/3\rangle_R^{\,2} &= \frac{\sinh\frac{\gamma_0+\gamma_{2\pi/3}}{2}\sinh\frac{\gamma_\pi+\gamma_{\pi/3}}{2}\sinh^2\frac{\gamma_{\pi/3}+\gamma_{2\pi/3}}{2}}{9\sinh\gamma_{2\pi/3}\sinh\gamma_{\pi/3}\sinh\frac{\gamma_0+\gamma_{\pi/3}}{2}\sinh\frac{\gamma_\pi+\gamma_{2\pi/3}}{2}},\end{aligned}$$

$$
{}_{NS}\langle\emptyset|\sigma|2\pi/3,-2\pi/3\rangle_R{}^2 = \frac{\sinh\frac{\gamma_0+\gamma_{\pi/3}}{2}\sinh\frac{\gamma_0+\gamma_{2\pi/3}}{2}}{12\sinh\gamma_{2\pi/3}\sinh\gamma_{\pi/3}\sinh\frac{\gamma_\pi+\gamma_{\pi/3}}{2}\sinh\frac{\gamma_\pi+\gamma_{2\pi/3}}{2}\sinh^2\frac{\gamma_{\pi/3}+\gamma_{2\pi/3}}{2}},
$$

$$
{}_{NS}\langle-\pi/3,\pi/3|\sigma|\emptyset\rangle_R{}^2 = \frac{\sinh\frac{\gamma_\pi+\gamma_{\pi/3}}{2}\sinh\frac{\gamma_\pi+\gamma_{2\pi/3}}{2}}{12\sinh\gamma_{2\pi/3}\sinh\gamma_{\pi/3}\sinh\frac{\gamma_0+\gamma_{\pi/3}}{2}\sinh\frac{\gamma_0+\gamma_{2\pi/3}}{2}\sinh^2\frac{\gamma_{\pi/3}+\gamma_{2\pi/3}}{2}},
$$

$$
{}_{NS}\langle\emptyset|\sigma|0,2\pi/3\rangle_R{}^2 = {}_{NS}\langle\emptyset|\sigma|0,-2\pi/3\rangle_R{}^2 = \frac{1}{12\sinh\frac{\gamma_0+\gamma_\pi}{2}\sinh\frac{\gamma_0+\gamma_{\pi/3}}{2}\sinh\frac{\gamma_\pi+\gamma_{\pi/3}}{2}\sinh\gamma_{\pi/3}},
$$

$$
{}_{NS}\langle-\pi/3,\pi|\sigma|\emptyset\rangle_R{}^2 = {}_{NS}\langle\pi/3,\pi|\sigma|\emptyset\rangle_R{}^2 = \frac{1}{12\sinh\frac{\gamma_0+\gamma_\pi}{2}\sinh\frac{\gamma_0+\gamma_{2\pi/3}}{2}\sinh\gamma_{2\pi/3}\sinh\frac{\gamma_\pi+\gamma_{2\pi/3}}{2}},
$$

$$
{}_{NS}\langle-\pi/3,\pi/3|\sigma|0,2\pi/3\rangle_R{}^2 = {}_{NS}\langle-\pi/3,\pi/3|\sigma|0,-2\pi/3\rangle_R{}^2 = \frac{4\sinh\frac{\gamma_0+\gamma_{\pi/3}}{2}\sinh\frac{\gamma_\pi+\gamma_{\pi/3}}{2}}{9\sinh\gamma_{\pi/3}\sinh\frac{\gamma_0+\gamma_\pi}{2}},
$$

$$
{}_{NS}\langle-\pi/3,\pi|\sigma|2\pi/3,-2\pi/3\rangle_R{}^2 = {}_{NS}\langle\pi/3,\pi|\sigma|2\pi/3,-2\pi/3\rangle_R{}^2 = \frac{4\sinh\frac{\gamma_0+\gamma_{2\pi/3}}{2}\sinh\frac{\gamma_\pi+\gamma_{2\pi/3}}{2}}{9\sinh\gamma_{2\pi/3}\sinh\frac{\gamma_0+\gamma_\pi}{2}},
$$

$$
{}_{NS}\langle-\pi/3,\pi|\sigma|0,2\pi/3\rangle_R{}^2 = {}_{NS}\langle\pi/3,\pi|\sigma|0,-2\pi/3\rangle_R{}^2 = \frac{1}{9},
$$

$$
{}_{NS}\langle-\pi/3,\pi|\sigma|0,-2\pi/3\rangle_R{}^2 = {}_{NS}\langle\pi/3,\pi|\sigma|0,2\pi/3\rangle_R{}^2 = \frac{4}{9}.
$$

This indeed can be done — with a little bit cumbersome but straightforward calculation[1]. We have performed a similar check for small N up to $N = 4$ and we have no doubt in the validity of (12) for arbitrary N. The rigorous proof of this formula will complete, in a sense, the study of the 2D Ising model in zero field.

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[1] Useful "building blocks", that greatly simplify it, can be found in the Appendix of [5].

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